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ON WARPED PRODUCT QUASI GENERALIZED RECURRENT MANIFOLDS

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Abstract: Recurrent manifold is one of the important curvature restricted geometric structure and quasi generalized recurrent manifold is a proper generalization of such manifolds. The object of the present paper is to study the properties of quasi generalized recurrent warped product manifolds and to determine the necessary and sufficient conditions for a warped product manifold to be a quasi generalized recurrent manifold. Finally as a support of the main result, we present an example of warped product quasi generalized recurrent manifold.

Keywords and Phrases: Recurrent manifold, quasi generalized recurrent manifold, recurrent like manifold, warped product manifold.

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1. Introduction

The notion of manifold of constant curvature is an important idea in differential geometry. In 1926, Cartan [4] generalized this geometric notion and introduced the concept of local symmetry and presented it as a curvature restriction $\nabla R = 0$, i.e., the Riemann-Christoffel curvature tensor R is covariantly constant. During the last eight decades, various authors are trying to generalize or to extend the idea of local symmetry by weakening its curvature restriction in different ways.

In 1946, Cartan [5] introduced the notion of semisymmetry, which is a generalization of local symmetry and later Szabó [33] also classified this notion of semisymmetry. Further generalizing this notion in 1983, Adamów and Deszcz [1] introduced the concept of pseudosymmetry, which is also known as Deszcz pseudosymmetry (see [18]). On the other hand, as a generalization of local symmetry, Chaki [6] introduced another notion of pseudosymmetry. It should be noted that the interrelation between two types of pseudosymmetries is studied by Shaikh et. al. [18]. In 1989, Tamássy and Binh [34] introduced the notion of weakly symmetric manifold which is a generalization of pseudosymmetry considered by Chaki. We note that various authors studied weakly symmetric manifold with various generalized curvature tensors (see [10], [19], [20], [21], [22], [23]). Recently, Shaikh and Kundu [24] obtained the characterization of warped product weakly symmetric manifold by generalizing the results of Binh [2].

Another generalization of local symmetry was introduced by Ruse ([14], [15], [16]) as κ-space, which was later renamed by Walker [35] as recurrent manifold. In 1979, Dubey [8] defined generalized recurrent manifold. But in 2012 [12], Olszak and Olszak showed that every generalized recurrent manifold is concircularly recurrent manifold and every concircularly recurrent manifold is again recurrent and hence every generalized recurrent manifold is a recurrent manifold. Again as a generalization of recurrent manifold, recently, Shaikh and his coauthors introduced four new types of generalized recurrent structures together with their proper existence, namely, quasi generalized recurrent manifold [29], hyper-generalized recurrent manifold [28], weakly generalized recurrent manifold [30] and super generalized recurrent manifold ([25], [31]). For the existence of such structures we refer the reader to see [17] and [32]. These kinds of generalizations of recurrent structure are known as recurrent like structures.

The main objective of the present paper is to investigate the characterization of quasi generalized recurrent warped product manifolds. We know that decomposable or product manifold is a special case of warped product manifold when the warping function is identically 1. Thus we can present the characterization of a decomposable manifold with quasi generalized recurrent structure.

The paper is organized as follows: Section 2 deals with rudimentary facts of various recurrent like structures. Section 3 is concerned with basic curvature relations of a warped product manifold. In Section 4 we present our main result and some corresponding corollaries, and finally in Section 5, a proper example of a warped product quasi generalized recurrent manifold is presented.

2. Preliminaries

Let M be a non-flat n-dimensional ($n \geq 3$) smooth manifold equipped with a semi-Riemannian metric g and we denote the corresponding Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature by ∇, R, S and κ respectively.
For two $(0,2)$-tensors A and E, their Kulkarni-Nomizu product [25], $A \wedge E$ is given by

$$(A \wedge E)_{ijkl} = A_{il}E_{jk} + A_{jk}E_{il} - A_{ik}E_{jl} - A_{jl}E_{ik}. \tag{2.1}$$

As its particular cases $g \wedge g$, $g \wedge S$ and $S \wedge S$ are respectively given by

$$(g \wedge g)_{ijkl} = 2(g_{il}g_{jk} - g_{ik}g_{jl}),$$

$$(g \wedge S)_{ijkl} = g_{il}S_{jk} + S_{il}g_{jk} - g_{ik}S_{jl} - S_{ik}g_{jl},$$

$$\text{and} \quad (S \wedge S)_{ijkl} = 2(S_{il}S_{jk} - S_{ik}S_{jl}).$$

Definition 2.2. *The n-dimensional manifold (M^n, g) is said to be recurrent* [35] *if*

$$R_{ijkl,m} = \Pi_m R_{ijkl} \tag{2.2}$$

holds on $\{x \in M : R \neq 0 \text{ at } x\}$ for an 1-form $\Pi \in \chi^*(M)$, where ',' denotes the covariant derivative. This 1-form Π is called the associated 1-form of the recurrent structure and such a manifold is denoted by K_n.

Definition 2.2. *The n-dimensional manifold (M^n, g) is said to be generalized recurrent (denoted by GK_n)* [8] *if*

$$R_{ijkl,m} = \Pi_m R_{ijkl} + \Theta_m (g_{il}g_{jk} - g_{ik}g_{jl}) \tag{2.3}$$

holds on $\{x \in M : R \neq 0 \text{ at } x\}$ for some 1-forms Π and Θ, where $G = \frac{1}{2} g \wedge g$ is the Gaussian curvature tensor.

In 2012, Olszak and Olszak [12] showed that every GK_n satisfying (2.3) is concircularly recurrent with

$$\nabla W = \Pi \otimes W \quad \text{(where } W \text{ is the concircular curvature tensor)}$$

and every concircularly recurrent manifold is again K_n with same associated 1-form and thus $\Theta = 0$. This shows that the structure GK_n reduces to K_n.

Definition 2.3. *The manifold (M^n, g) is said to be quasi generalized recurrent ([29], [31]) if the following condition*

$$R_{ijkl,m} = \Pi_m R_{ijkl} + \Phi_m[g_{il}g_{jk} - g_{ik}g_{jl}] + \Psi_m[g_{il}\eta_j\eta_k + g_{jk}\eta_i\eta_l - g_{ik}\eta_j\eta_l - g_{jl}\eta_i\eta_k], \quad (2.4)$$

holds on $\{x \in M : R \neq 0$ and $g \wedge (\eta \otimes \eta) \neq 0$ at $x\}$ for some Π, Φ, Ψ and $\eta \in \chi^*(M)$, called the associated 1-forms.

Our main focus in this paper is on the QGK_n-structure. We note that the defining condition of a QGK_n was first considered by Shaikh and Roy [29] as

$$R_{ijkl,m} = \Pi_m R_{ijkl} + \Phi_m[g_{il}g_{jk} - g_{ik}g_{jl} + g_{il}\eta_j\eta_k + g_{jk}\eta_i\eta_l - g_{ik}\eta_j\eta_l - g_{jl}\eta_i\eta_k]. \quad (2.5)$$

Definition 2.4. *The Riemannian manifold (M^n, g) is said to be of quasi constant curvature [7] if the Riemannian curvature tensor R of type (0, 4) satisfies the following condition*

$$R = L_1 G + L_2[g \wedge (\eta \otimes \eta)], \quad (2.6)$$

where L_1, L_2 are scalar functions and η is a non-zero 1-form. Moreover, if $L_2 = 0$, then the manifold becomes a manifold of constant curvature.

3. Warped Product Manifolds

Let $(\overline{M}, \overline{g})$ and $(\widetilde{M}, \widetilde{g})$ be two semi-Riemannian manifolds of dimension p and $(n-p)$ respectively $(1 \leq p < n)$, and f is a positive smooth function on $\overline{M}$. Then the warped product $M = \overline{M} \times_f \widetilde{M}$ is the product manifold ([3], [11]) $\overline{M} \times \widetilde{M}$ of dimension n endowed with the metric

$$g = \pi^*(\overline{g}) + (f \circ \pi)\sigma^*(\widetilde{g}), \quad (3.1)$$

where $\pi : M \to \overline{M}$ and $\sigma : M \to \widetilde{M}$ are the natural projections. Then the components of the warped product metric g are given by

$$g_{ij} = \begin{cases} \overline{g}_{ij} & \text{for } i = a \text{ and } j = b, \\ f\widetilde{g}_{ij} & \text{for } i = \alpha \text{ and } j = \beta, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

Here $a, b \in \{1, 2, ..., p\}$ and $\alpha, \beta \in \{p+1, p+2, ..., n\}$. Throughout the paper, we will consider $a, b, c, ... \in \{1, 2, ..., p\}$ and $\alpha, \beta, \gamma, ... \in \{p+1, p+2, ..., n\}$ and $i, j, k, ... \in \{1, 2, ..., n\}$.

It should be mentioned that $\overline{M}$ is called the base, $\widetilde{M}$ is called the fiber and f is called the warping function. Again we assume that, when Ω is a quantity formed with respect to g, we denote by $\overline{\Omega}$ and $\widetilde{\Omega}$, the similar quantities formed with respect to $\overline{g}$ and $\widetilde{g}$ respectively.

The non-zero components of Levi-Civita connection ∇ of M are given by

$$\Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma}, \qquad \Gamma^a_{\beta\gamma} = -\frac{1}{2}\overline{g}^{ab} f_b \widetilde{g}_{\beta\gamma}, \quad \Gamma^\alpha_{a\beta} = \frac{1}{2f} f_a \delta^\alpha_\beta, \tag{3.3}$$

where $f_a = \partial_a f = \frac{\partial f}{\partial x^a}$.

The components of the Riemann-Christoffel curvature tensor R and Ricci tensor S of M which may not vanish identically are the followings:

$$R_{abcd} = \overline{R}_{abcd}, \quad R_{a\alpha b\beta} = f T_{ab} \widetilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = f \widetilde{R}_{\alpha\beta\gamma\delta} - f^2 P \widetilde{G}_{\alpha\beta\gamma\delta}, \tag{3.4}$$

$$S_{ab} = \overline{S}_{ab} - (n-p) T_{ab}, \quad S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} + Q \widetilde{g}_{\alpha\beta}, \tag{3.5}$$

where $G_{ijkl} = \frac{1}{2}(g \wedge g)_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}$ are the components of Gaussian curvature and

$$T_{ab} = -\frac{1}{2f}\left(\nabla_b f_a - \frac{1}{2f} f_a f_b\right), \qquad tr(T) = g^{ab} T_{ab},$$

$$Q = f[(n-p-1)P - tr(T)], \qquad P = \frac{1}{4f^2} g^{ab} f_a f_b.$$

Again the non-zero components of ∇R are given by [9]:

$$\begin{cases} (i) R_{abcd,e} = \overline{R}_{abcd,e}, \\ (ii) R_{a\alpha b\beta,e} = f T_{ab,e} \widetilde{g}_{\alpha\beta}, \\ (iii) R_{\alpha\beta\gamma\delta,e} = -f_e \widetilde{R}_{\alpha\beta\gamma\delta} + f^2 P_e \widetilde{G}_{\alpha\beta\gamma\delta}, \\ (iv) R_{\alpha\beta\gamma\delta,\epsilon} = f \widetilde{R}_{\alpha\beta\gamma\delta,\epsilon}, \\ (v) R_{\alpha\beta\gamma d,\epsilon} = -\frac{f_d}{2} \widetilde{R}_{\alpha\beta\gamma\epsilon} + \frac{f^2}{2} P_d \widetilde{G}_{\alpha\beta\gamma\epsilon}, \\ (vi) R_{abc\delta,\epsilon} = \frac{1}{2}\widetilde{g}_{\epsilon\delta}(f_a T_{bc} - f_b T_{ac}) + \frac{1}{2} f^d \overline{R}_{abcd} \widetilde{g}_{\epsilon\delta}. \end{cases} \tag{3.6}$$

The non-zero components of the Gaussian curvature tensor G are given by

$$\begin{cases} (i) G_{abcd} = \overline{G}_{abcd}, \\ (ii) G_{a\alpha b\beta} = -f \overline{g}_{ab} \widetilde{g}_{\alpha\beta}, \\ (iii) G_{\alpha\beta\gamma\delta} = f^2 \widetilde{G}_{\alpha\beta\gamma\delta}. \end{cases} \tag{3.7}$$

For detailed information about the components of various tensors on a warped product manifold, we refer the reader to see [13], [24], [26], [27] and also the references therein.

4. Warped Product QGK_n

Theorem 4.1. *Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product manifold. Then M is a QGK_n with (Π, Φ, Ψ, η) if and only if the following conditions hold simultaneously:*

$$\begin{cases} (i) \quad \overline{\nabla}\overline{R} = \overline{\Pi} \otimes \overline{R} + \overline{\Phi} \otimes \overline{G} + \overline{\Psi} \otimes \overline{g} \wedge (\overline{\eta} \otimes \overline{\eta}), \\ (ii) \quad \widetilde{\Pi} \otimes \overline{R} + \widetilde{\Phi} \otimes \overline{G} + \widetilde{\Psi} \otimes \overline{g} \wedge (\overline{\eta} \otimes \overline{\eta}) = 0, \end{cases} \tag{4.1}$$

$$\begin{cases} (iii) \quad (df + f\overline{\Pi}) \otimes \widetilde{R} = f^2(dP + P\overline{\Pi} - \overline{\Phi}) \otimes \widetilde{G} - f\overline{\Psi} \otimes \widetilde{g} \wedge (\widetilde{\eta} \otimes \widetilde{\eta}), \\ (iv) \quad \widetilde{\nabla}\widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R} - f(P\widetilde{\Pi} - \widetilde{\Phi}) \otimes \widetilde{G} + \widetilde{\Psi} \otimes \widetilde{g} \wedge (\widetilde{\eta} \otimes \widetilde{\eta}), \end{cases} \tag{4.2}$$

$$\begin{cases} (v) \ \nabla T \otimes \widetilde{g} = (\overline{\Pi} \otimes T - \overline{\Phi} \otimes \overline{g} - \overline{\Psi} \otimes \overline{\eta} \otimes \overline{\eta}) \otimes \widetilde{g} - \frac{1}{f}\overline{\Psi} \otimes \overline{g} \otimes (\widetilde{\eta} \otimes \widetilde{\eta}), \\ (vi) \ (T \otimes \widetilde{\Pi} - \overline{g} \otimes \widetilde{\Phi}) \otimes \widetilde{g} = (\overline{\eta} \otimes \overline{\eta}) \otimes \widetilde{\Psi} \otimes \widetilde{g} + \frac{1}{f}\overline{g} \otimes \widetilde{\Psi} \otimes (\widetilde{\eta} \otimes \widetilde{\eta}), \end{cases} \tag{4.3}$$

$$\begin{cases} (vii) \quad f^d \overline{R}_{abcd} = -(f_a T_{bc} - f_b T_{ac}), \\ (viii) \quad df \otimes \widetilde{R} = f^2 dP \otimes \widetilde{G}. \end{cases} \tag{4.4}$$

Proof. First suppose that M is a QGK_n. Then in terms of local coordinates, the defining condition can be written as

$$R_{ijkl,m} = \Pi_m R_{ijkl} + \Phi_m G_{ijkl} + \Psi_m (g \wedge (\eta \otimes \eta))_{ijkl}. \tag{4.5}$$

Putting

$$\begin{cases} (i) \quad i = a, j = b, k = c, l = d, m = e \ \text{ and} \\ (ii) \quad i = a, j = b, k = c, l = d, m = \epsilon \end{cases}$$

respectively in (4.5) and then in view of (3.4)-(3.7) it is easy to check that (4.1) holds. Similarly putting

$$\begin{cases} (iii) \quad i = \alpha, j = \beta, k = \gamma, l = \delta, m = e; \\ (iv) \quad i = \alpha, j = \beta, k = \gamma, l = \delta, m = \epsilon; \\ (v) \quad i = a, j = \alpha, k = b, l = \beta, m = e; \\ (vi) \quad i = a, j = \alpha, k = b, l = \beta, m = \epsilon; \\ (vii) \quad i = a, j = b, k = c, l = \alpha, m = \epsilon \ \text{ and} \\ (viii) \quad i = \alpha, j = \beta, k = \gamma, l = a, m = \epsilon \end{cases}$$

respectively in (4.5) and then in view of (3.4)-(3.7) we get (4.2)-(4.4) respectively. The converse part is obvious. This proves the theorem.

From Theorem 4.1, it follows that the nature of base and fiber of a warped product QGK_n are given by the following:

Corollary 4.1. *Let* $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ *be a warped product* QGK_n *with* (Π, Φ, Ψ, η). *Then*
(*i*) $\overline{M}$ *is a* QGK_p *with* $(\overline{\Pi}, \overline{\Phi}, \overline{\Psi}, , \overline{\eta})$.
(*ii*) $\overline{M}$ *is of quasi constant curvature on* $\{x \in M : \widetilde{\Pi}_x \neq 0\}$.
(*iii*) $\widetilde{M}$ *is a* QGK_{n-p} *with* $(\widetilde{\Pi}, f\widetilde{\Phi} - fP\widetilde{\Pi}, \widetilde{\Psi}, \widetilde{\eta})$.
(*iv*) $\widetilde{M}$ *is of quasi constant curvature on* $\{x \in M : (df + f\overline{\Pi})_x \neq 0\}$.
(*v*) $\widetilde{M}$ *is of constant curvature on* $\{x \in M : df_x \neq 0\}$.

From Theorem 4.1, the characterization of a decomposable or product QGK_n is given by the following:

Corollary 4.2. *Let* $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ *be a product manifold. Then* M *is a* QGK_n *with* (Π, Φ, Ψ, η) *if and only if the following conditions simultaneously hold*

$$\begin{cases} (i) & \overline{\nabla} \overline{R} = \overline{\Pi} \otimes \overline{R} + \overline{\Phi} \otimes \overline{G} + \overline{\Psi} \otimes \overline{g} \wedge (\overline{\eta} \otimes \overline{\eta}), \\ (ii) & \widetilde{\Pi} \otimes \overline{R} + \widetilde{\Phi} \otimes \overline{G} + \widetilde{\Psi} \otimes \overline{g} \wedge (\overline{\eta} \otimes \overline{\eta}) = 0, \end{cases}$$

$$\begin{cases} (iii) & \overline{\Pi} \otimes \widetilde{R} = -\overline{\Phi} \otimes \widetilde{G} - \overline{\Psi} \otimes \widetilde{g} \wedge (\widetilde{\eta} \otimes \widetilde{\eta}), \\ (iv) & \widetilde{\nabla} \widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R} + \widetilde{\Phi} \otimes \widetilde{G} + \widetilde{\Psi} \otimes \widetilde{g} \wedge (\widetilde{\eta} \otimes \widetilde{\eta}), \end{cases}$$

$$\begin{cases} (v) & (\overline{\Phi} \otimes \overline{g} + \overline{\Psi} \otimes \overline{\eta} \otimes \overline{\eta}) \otimes \widetilde{g} + \overline{\Psi} \otimes \overline{g} \otimes (\widetilde{\eta} \otimes \widetilde{\eta}) = 0, \\ (vi) & (\overline{g} \otimes \widetilde{\Phi}) \otimes \widetilde{g} + (\overline{\eta} \otimes \overline{\eta}) \otimes \widetilde{\Psi} \otimes \widetilde{g} + \overline{g} \otimes \widetilde{\Psi} \otimes (\widetilde{\eta} \otimes \widetilde{\eta}) = 0. \end{cases}$$

Again as special case of the Theorem 4.1, we can state the following corollary:

Corollary 4.3. [24] *Let* $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ *be a warped product manifold. Then* M *is a recurrent manifold with*

$$\nabla R = \Pi \otimes R$$

if and only if the following conditions simultaneously hold
1.(*i*) $\overline{\nabla} \overline{R} = \overline{\Pi} \otimes \overline{R}$, (*ii*) $\widetilde{\Pi} \otimes \overline{R} = 0$,
2.(*i*) $-(df + f\overline{\Pi}) \otimes \widetilde{R} = \frac{1}{2} f^2 (P\widetilde{\Pi} - dP) \otimes \widetilde{g} \wedge \widetilde{g}$, (*ii*) $\widetilde{\nabla} \widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R}$, $P\widetilde{\Pi} = 0$,
3.(*i*) $\overline{\nabla} T = \overline{\Pi} \otimes T$, (*ii*) $\widetilde{\Pi} \otimes T = 0$,
4.(*i*) $f^d \overline{R}_{abcd} = -(f_a T_{bc} - f_b T_{ac})$ *and* (*ii*) $df \otimes \widetilde{R} = f^2 dP \otimes \widetilde{G}$.

Corollary 4.4. *If* $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ *is a warped product recurrent manifold with*

associated 1-form Π, *then*
(*i*) $\overline{M}$ *and* $\widetilde{M}$ *are both recurrent. Also* T *is recurrent with associated 1-form* Π.
(*ii*) $\overline{R} = 0$, $T = 0$ *and* $P = 0$ *on the set* $\{x \in M : \widetilde{\Pi} \neq 0\}$.
(*iii*) $\widetilde{M}$ *is of constant curvature on* $\{x \in M : df_x \neq 0\} \cup \{x \in M : (df + f\overline{\Pi})_x \neq 0\}$.

Corollary 4.5. *Let* $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ *be a product manifold. Then* M *is a recurrent manifold with associated 1-form* Π *if and only if the following conditions simultaneously hold*
(*i*) $\overline{\nabla R} = \overline{\Pi} \otimes \overline{R}$, $\quad \widetilde{\Pi} \otimes \overline{R} = 0$, $\quad$ (*ii*) $\overline{\Pi} \otimes \widetilde{R} = 0$, $\quad \widetilde{\nabla} \widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R}$.

5. Example

Now in this section we establish the existence of warped product QGK_n. For this purpose we take a 3-dimensional recurrent manifold of quasi-constant curvature as base and a 1-dimensional fiber.

Example 5.1. Consider the warped product manifold $M = \overline{M} \times_f \widetilde{M}$, where $\overline{M}$ is a 3-dimensional manifold equipped with the metric

$$\overline{ds}^2 = e^{x^1+x^3}(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2$$

in local coordinates (x^1, x^2, x^3) and $\widetilde{M}$ is an open interval of $\mathbb{R}$ with local coordinate x^4 and the warping function $f = e^{x^1}$. Then the metric of the 4-dimensional warped product manifold M is given by

$$ds^2 = g_{ij}dx^idx^j = e^{x^1+x^3}(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + e^{x^1}(dx^4)^2. \tag{5.1}$$

The non-zero components (up to symmetry) of the Riemann-Christoffel curvature tensor R are given by

$$R_{1313} = -\frac{1}{2}e^{x^1+x^3}, \quad R_{1414} = -\frac{e^{x^1}}{4}. \tag{5.2}$$

The non-zero components (up to symmetry) of ∇R are given by:

$$R_{1313,1} = -\frac{e^{x^1+x^3}}{2} = R_{1313,3}. \tag{5.3}$$

Now using the components of R, G and ∇R, it is easy to check that the manifold M is $(QGK)_4$ with (Π, Φ, Ψ, η) for $\Phi \equiv 0$,

$$\Pi = \left\{ \frac{2e^{x^1+x^3}}{2e^{x^1+x^3} - 1}, 0, \frac{2e^{x^1+x^3}}{2e^{x^1+x^3} - 1}, 0 \right\},$$

$$\Psi = \left\{ \frac{-2e^{x^1+x^3}}{(2e^{x^1+x^3}-1)(2e^{x^1+x^3}+1)}, 0, \frac{-2e^{x^1+x^3}}{(2e^{x^1+x^3}-1)(2e^{x^1+x^3}+1)}, 0 \right\}$$

and

$$\eta = \left\{ -\frac{1}{2}\sqrt{1+2e^{x^1+x^3}}, 0, 0, 0 \right\}.$$

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